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# Tendency to occupy a statistically dominant spatial state of the flow as a driving force for turbulent transition

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**Abstract** A simple analytical model for a turbulent flow is proposed, which considers the flow as a collection of localized spatial structures that are composed of elementary "cells" in which the state of the particles (atoms or molecules) is uncertain. The Reynolds number is associated with the ratio between the total phase volume for the system and that for the elementary cell. Calculating the statistical weights of the collections of the localized structures, it is shown that as the Reynolds number increases, the elementary cells group into the localized structures, which successfully explains the onset of turbulence and some other characteristic properties of turbulent flows. It is also shown that the basic assumptions underlying the model are involved in the derivation of the Navier-Stokes equation, which suggests that the driving force for the turbulent transition described with the hydrodynamic equations is essentially the same as in the present model, i.e. the tendency of the system to occupy a statistically dominant state plays a key role. The instability of the flow can then be a mechanism to initiate the structural rearrangement of the flow to find this state.

**Keywords** Fluid flow · Transition to turbulence · Driving force · Statistical model

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**Mathematics Subject Classification (2000)** 76F02 · 76F06 · 76F55

## 1 Introduction

The transition from laminar to turbulent fluid motion occurring at large Reynolds numbers [1,2,3,4,5] is generally associated with the instability of the laminar flow, and this viewpoint is well supported by the analysis of solutions of hydrodynamic equations [4,5,6]. On the other hand, since the turbulent flow characteristically appears in the form of eddies filling the flow field, the tendency to occupy such a structured state of the flow cannot be ruled out as a driving force for turbulent transition. To examine this possibility, we propose a simple analytical model for the flow and show that as the Reynolds number increases, the state of the flow in the form of collection of localized spatial structures (eddies) becomes statistically more favorable than the unstructured state, which successfully explains the onset of turbulence and some other general properties of turbulent flows. We also show that the basic assumptions underlying the model are involved in the derivation of the Navier-Stokes equation, which suggests that the driving force for the turbulent transition is basically the same as in the present model, i.e. the tendency of the system to occupy a statistically dominant state plays a key role. The

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instability of the flow at high Reynolds numbers can then be a mechanism to initiate the structural rearrangement of the flow to find this state.

## 2 Model

Let a fluid flow be represented by a system of  $N$  identical particles (atoms or molecules). Assume that the particles can form localized structures of  $N_i$  particles (with the number of such structures being  $M_i$ ), which are similar to turbulent eddies in that the particles execute an overall concerted motion. Since the concerted motion should break down at the microscale level, each  $i$  structure can be divided into  $N_i/n$  elementary "cells" (each of  $n$  particles), in which the states of the particles (positions and velocities) are uncorrelated. Then, considering the interaction between the localized structures to be small and taking into account that the permutations of the particles in the elementary cells, those of the elementary cells in the localized structures, and the permutations of the localized structures themselves do not lead to new states, the total number of states can be estimated as for an ideal-gas system [7]

$$\Gamma = \frac{N!}{\prod_i [(n!)^{N_i/n} (N_i/n)!]^{M_i} M_i!}$$

or, with the Stirling approximation  $x! \approx (x/e)^x$  to be applicable

$$\Gamma = \frac{(N/n)^N e^{N/n}}{\prod_i (N_i/n)^{N_i M_i/n} (M_i/e)^{M_i}} \quad (1)$$

where  $\sum_i N_i M_i = N$ .

With Eq. (1), it is possible to calculate the most probable size distribution of the structures,  $\tilde{M}_i = \tilde{M}_i(N_i)$ , which maximizes the number of states  $\Gamma$  and, correspondingly, the entropy  $S = \ln \Gamma$ . This distribution reduces the phase space of the system to a subspace of much lower dimension, similar to how, according to the dynamical system theory, the phase space shrinks when the system reaches the (strange) attractor in the transition to turbulence [4, 8]. Applying the Lagrange multiplier method to the entropy functional, i.e. varying the expression  $\sum_i [-(M_i N_i/n) \ln(N_i/n) - M_i \ln(M_i/e)] + \alpha' \sum_i M_i N_i + \beta' \sum_i M_i E_i$  with respect to  $M_i$  at two conservation conditions  $\sum_i M_i N_i = N$  and  $\sum_i M_i E_i = E$  ( $E_i$  is the kinetic energy of fluctuations in  $i$  structure, and  $E$  is the total kinetic energy), one obtains

$$\tilde{M}_i = \left( \frac{N_i}{n} \right)^{-\frac{N_i}{n}} e^{\alpha' N_i + \beta' E_i} \quad (2)$$

where  $\alpha'$  and  $\beta'$  are the Lagrange multipliers.

The dependence of  $E_i$  on  $N_i$  is specific rather than universal; it is clearly different, e.g., for homogeneous and shear-layer turbulence. To be definite, assume that the velocity of fluctuations increases with the distance  $r$  as in the Kolmogorov theory of turbulence for the inertial range of scales [4, 9, 10], i.e.

$$v(r) \sim r^h \quad (3)$$

where  $h = 1/3$ . Then, the kinetic energy of fluctuations per unit mass is  $e(r) \sim r^{2h}$ , and  $E_i \sim \int_0^{r_i} e(r) r^2 dr \sim r_i^{2h+3} \sim N_i^{2h/3+1}$ . Also, it is convenient to pass from  $N_i$  to the number of the elementary cells in the localized structure  $q_i = N_i/n$ . With these changes, Eq. (2) becomes

$$\tilde{M}_i = q_i^{-q_i} e^{\alpha q_i + \beta q_i^{2h/3+1}} \quad (4)$$

where  $\alpha$  and  $\beta$  are new constants that are determined from the equations of conservation of the total number of particles and kinetic energy

$$N/n = \sum_i \tilde{M}_i q_i \quad (5)$$

and

$$E/n^{2h/3+1} = \sum_i \tilde{M}_i q_i^{2h/3+1} \quad (6)$$

Given the values of  $n$ ,  $N$  and  $E$ , the constants  $\alpha$  and  $\beta$  can be calculated by generating a set of  $q_i$  and solving Eqs. (5) and (6) with  $\tilde{M}_i$  substituted from Eq. (4). Another possibility is to vary  $\alpha$  and  $\beta$  to obtain  $N/n$  and  $E/n^{2h/3+1}$  as functions of  $\alpha$  and  $\beta$ .

The parameter  $N/n$  can be associated with the Reynolds number  $\text{Re}_L = VL/\nu$ , where  $V$  and  $L$  are, respectively, the velocity and linear scale characterizing the system as a whole, and  $\nu$  is the kinematic viscosity of the fluid. Since the particles are identical, the  $6N$ -dimensional phase space is reduced to the 6-dimensional single-particle space. Then, the linear size of the elementary cell, in which the behavior of the particles is uncorrelated, can be taken as the product of the characteristic velocity and length at which the state of the particles is uncertain. For a gas fluid, it is the product of the molecular thermal velocity  $c$  and the mean free path  $\lambda$  [11]. Then, the phase volume of the cell is  $(c\lambda)^3$ , with the number of particles in the cell being proportional to this volume,  $n \sim (c\lambda)^3$ . Correspondingly, the total number of particles  $N$  can be considered to be proportional to the total phase volume for the system  $(VL)^3$ , i.e.  $N \sim (VL)^3$  (the density of the fluid is assumed to be constant). Taking into account the kinetic theory expression for the gas viscosity  $\nu \sim c\lambda$  (e.g., Ref. [12]), one obtains

$$N/n \sim (VL/\nu)^3 = \text{Re}_L^3 \quad (7)$$

A similar relation is valid for a liquid fluid. Here, the linear size of the elementary cell can be taken to be  $c^2\tau$ , where  $c^2$  presents the fluctuations of the (molecular) kinetic energy per unit mass, and  $\tau$  is the mean residence time of the molecule in a settled state. Since the viscosity of liquid is  $\nu \sim c^2\tau$  (e.g., Ref. [13]), we arrive at Eq. (7) again. In what follows, we will assume for simplicity that the coefficient of proportionality in Eq. (7) is equal to 1, i.e.

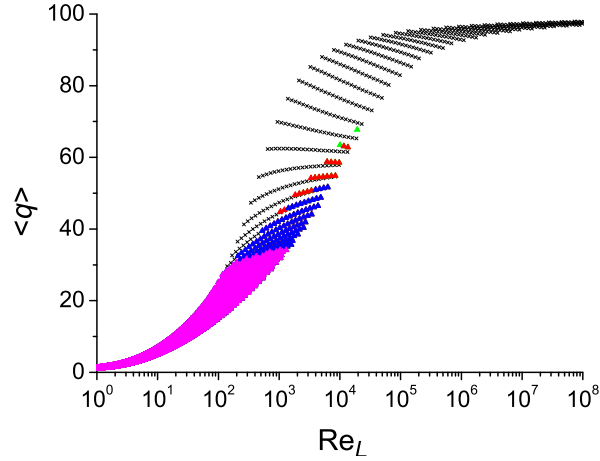
$$\text{Re}_L = (N/n)^{1/3} \quad (8)$$

### 3 Results and Discussion

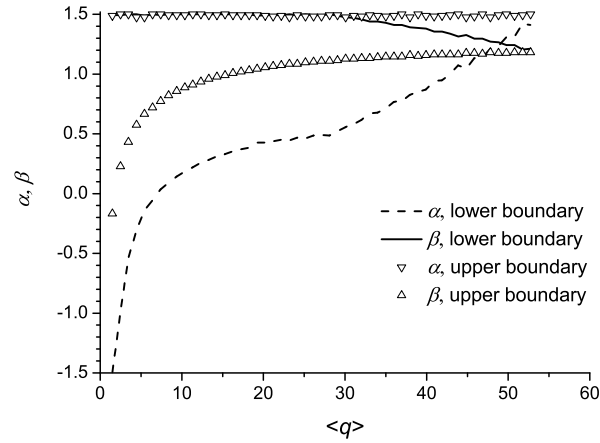
#### 3.1 Turbulent Transition Region

Figure 1 shows a characteristic dependence of the average number of elementary cells in the localized structures  $\langle q \rangle = \sum_i q_i \tilde{M}_i / \sum_i \tilde{M}_i$  on the Reynolds number  $\text{Re}_L$ ; the number of elementary cells  $q_i$  varies from 1 to  $q_{\max}$ . Points labelled with crosses represent all combinations of  $\alpha$  and  $\beta$ , with these parameters varied independently within  $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$  and  $\beta_{\min} \leq \beta \leq \beta_{\max}$ . However, not every combination leads to a physically reasonable bell-shaped distribution [14] (see also Fig. 4 below). In small fraction of the combinations (typically at large values of  $\beta$ ),  $\tilde{M}_i$  does not vanish at  $q \rightarrow q_{\max}$ . The fraction of such "wrong" combinations is generally within several percentages (in the given case, below 2 percent). Triangles show the dependence  $\langle q \rangle$  on  $\text{Re}_L$  with those "wrong"  $\alpha/\beta$  combinations excluded. The increase of  $q_{\max}$  does not change the distribution, except that the range of variation of  $\langle q \rangle$  and  $\text{Re}_L$  is extended to larger values of these quantities.

It is seen that as  $\text{Re}_L$  exceeds some characteristic value ( $\text{Re}_L \sim 10^2$  in Fig. 1),  $\langle q \rangle$  rapidly increases, i.e. the elementary cells group into localized structures, signaling that the flow becomes turbulent at these Reynolds numbers. It is noteworthy that the dependence of  $\langle q \rangle$  on  $\text{Re}_L$  is not unique, i.e. the same values of  $\langle q \rangle$  are observed in a broad range of  $\text{Re}_L$ . This indicates that the turbulent state is not solely determined by the Reynolds number; rather, as is well-known, it is flow specific, depending on the type of the flow, the inlet conditions, the flow environment, etc. [3,4,5,6]. The present statistical model is too simple to take these effects into account, but it offers an estimate for the range of  $\text{Re}_L$  at which the turbulent motion can be expected, i.e. the lower and upper bounds of the turbulent transition region. The lower and upper boundaries of the manifold of the "correct" points in Fig. 1 (labelled with triangles) are determined, respectively, by the maximum values of  $\beta$  and  $\alpha$  at which the correct points are obtained (Fig. 2). As these parameters increase, the lower boundary shifts to smaller values of  $\text{Re}_L$ , and the upper boundary to larger values of  $\text{Re}_L$  (Supplementary Material). More specifically, at the lower boundary  $\text{Re}_L \sim \exp(-0.15\beta\langle q \rangle)$ , and at the upper boundary  $\text{Re}_L \sim \exp(0.06\alpha\langle q \rangle)$ . Figure 3 shows these dependencies for  $\langle q \rangle = 40$ , which corresponds to approximately the maxima of the size distributions of the localized structures (see Fig. 4 and its discussion below). It is remarkable that the lower boundary moves until  $\text{Re}_L \sim 10^2$  is reached; after that it "freezes" (Supplementary Material). The model thus predicts that the flow should be laminar at  $\text{Re}_L < \text{Re}_L^*$  ( $\text{Re}_L^* \sim 10^2$  in the present



**Fig. 1** (Color online) Average number of the elementary cells in the localized structures  $\langle q \rangle$  as a function of  $Re_L$ . The number of the elementary cells  $q_i$  varies from 1 to  $q_{\max}$ , and parameters  $\alpha$  and  $\beta$  vary independently from  $\alpha_{\min} = \beta_{\min} = -1.5$  to  $\alpha_{\max} = \beta_{\max} = 1.5$  with the interval 0.015. Crosses represent all combinations of  $\alpha$  and  $\beta$  at  $q_{\max} = 100$ , and the triangles stand for the combinations which lead to the physically reasonable bell-shaped distribution at different  $q_{\max}$ ; the magenta, blue, red and green triangles are for  $q_{\max} = 100, 200, 300$  and 400, respectively.



**Fig. 2** The values of the parameters  $\alpha$  and  $\beta$  at the lower and upper boundaries of the transition region of Fig. 1;  $q_{\max} = 200$ .

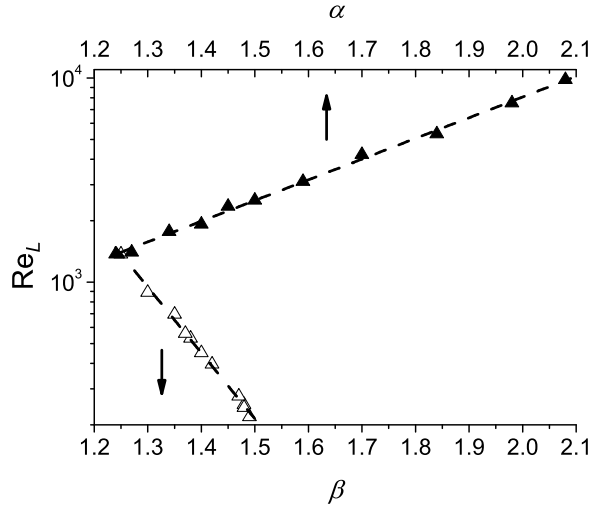
model), and it can remain laminar up to very large values of  $Re_L$ . This prediction is in good general agreement with the experimental and simulation results [3,4,5]. In particular, for the pipe flow, in his seminal work Reynolds [1] has found that the critical Re number varied from  $Re_L \approx 2 \times 10^3$  to  $Re_L \approx 1.3 \times 10^4$  depending on the inlet conditions. The lower bound of the stability of the laminar flow, which is observed at large disturbances of the inlet flow, has been confirmed in many experimental and theoretical works (e.g., Refs. [15] and [16], respectively), and the upper bound, which is achieved in carefully controlled conditions, has been found as high as  $Re_L \approx 1 \times 10^5$  [17]. In the present model, the above mentioned lower bound of the stability ( $Re_L \sim 10^3$ ) corresponds to  $\beta_{\max} \approx 1.25$ , and the upper bound is unlimited (Fig. 3).

### 3.2 Other Characteristic Properties of Turbulent Flows

The model is also consistent with some other characteristic properties of turbulent flows:

- The dissipation scale  $\eta$  obeys the known equation in the Kolmogorov theory of turbulence

$$\eta/L \sim Re_L^{-3/4} \quad (9)$$



**Fig. 3** The Reynolds number  $Re_L$  corresponding to the lower (open triangles) and the upper (solid triangles) boundaries of the transition range as a function of parameters  $\beta$  and  $\alpha$ , respectively;  $\langle q \rangle = 40$ . The dashed lines are the best exponential fits, see the text.

Indeed, similarly to  $(N/n)^{1/3}$  in Eq. (8), the quantity  $q_i^{1/3} = (N_i/n)^{1/3}$  can be considered to be the Reynolds number of  $i$  structure, i.e.  $Re_i = q_i^{1/3}$ . According to the Kolmogorov theory [9],  $Re_i \sim 1$  characterizes the dissipation scale  $\eta$ . Let us rewrite  $q_i = N_i/n$  as  $(N_i/N) \times (N/n)$  and take into account that  $v_i/V \sim (l_i/L)^{1/3}$  [Eq. (3)], so that  $N_i/N \sim [v_i l_i / (VL)]^3 \sim (l_i/L)^4$ . Then,

$$q_i \sim (N/n)(l_i/L)^4 = Re_L^3 (l_i/L)^4 \quad (10)$$

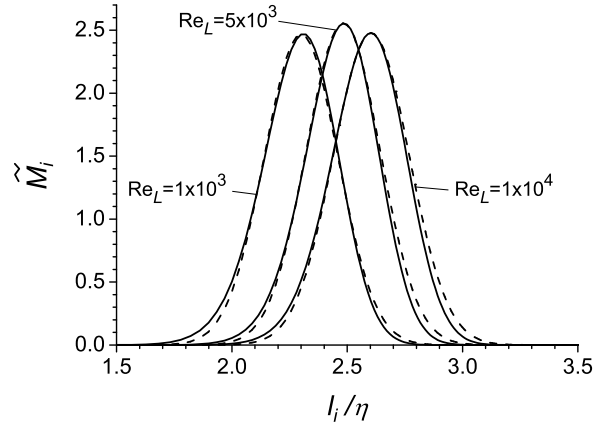
and the equality  $q_i \sim 1$  (with  $l_i \equiv \eta$ ) leads to Eq. (9). This, in particular, suggests that the linear size of the elementary cell could also be determined as  $\eta v_\eta$ , where  $v_\eta$  is the velocity increment at the dissipation scale.

- The probability density distributions of the linear sizes of the structures  $\tilde{M}_i(l_i/\eta)$ , Fig.4, are in general agreement with the results of the direct numerical simulation of isotropic homogeneous turbulence by Jiménez *et al.* [14], who found that the vortex radius distributions are bell-shaped, practically do not shift with  $Re_L$  (with the radius  $r$  measured in  $\eta$  units), and have maxima at  $r \approx 3\eta$ . According to Eqs. (9) and (10),  $l_i/\eta \approx (q_i)^{1/4}$ , so that the maxima of the distributions at  $l_i/\eta \approx 2.5$  correspond to  $q_i \approx 40$ .

- The structure functions and energy spectrum are similar to those in the Kolmogorov-Obukhov theory [9,18]. Since the dependence  $v = v(r)$  is assumed to be the same for all localized structures, the structure functions are independent of the size distributions of localized structures. With Eq. (3), the structure function of  $p$ -th order is  $S_p(l) = \langle \delta v_l^p \rangle \sim l^{p/3}$ , where  $l$  is the space increment, and the energy spectrum is  $E(k) \sim k^{-5/3}$ , where  $k = 2\pi/l$  is the wave number.

### 3.3 Connection with the Hydrodynamic Equations

The above consideration shows that the proposed model captures characteristic properties of turbulent flows, in particular, those obtained with the hydrodynamic equations. Then, the natural question to ask is: Why do two apparently different approaches, the hydrodynamic equations and the present model, lead to similar results? Or, what is common to these approaches to lead to similar results? The present model is based on three assumptions: i) the particles are identical, ii) there exists an elementary cell of the phase space in which the state of the particles is uncertain, and iii) the kinetic energy of the fluctuations is as in the Kolmogorov theory for the inertial interval of scales. The latter assumption seems to be not very essential for our purpose. For example, even if a quasi-solid rotation of the localized structures (eddies) is supposed, which highly overestimates their kinetic energy because the absence of the decrease of the velocity outside the eddy core [19], the statistical preference of the



**Fig. 4** Characteristic linear size distributions of the localized structures,  $1 \leq q_i \leq 200$ . Solid lines show the  $\tilde{M}_i(l_i/\eta)$  distributions averaged over all combinations of  $\alpha$  and  $\beta$  for which  $Re_L$  are close to those indicated at the curves within 1 percent (typically one of such distributions dominated). The dashed curves are the Gaussian fits to the distributions.

structured flow is still preserved (Supplementary Material). The attention should thus be focused on the former two assumptions. As well known, the Navier-Stokes equation for a gas fluid can be derived from the Boltzmann equation (the Chapman-Enskog expansion [12]), which, in turn, can be obtained from the Liouville equation (see, e.g., Ref. [20]). In the transition from the Liouville equation to the Boltzmann equation, first, the many-particle distribution function is reduced to the single-particle one under the assumption that the particles are identical (the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy, see, e.g. Ref. [21]), and, secondly, because the gas density is taken to be low, the interaction between the particles is assumed to be a random two-body collision, so that the positions and velocities of the particles are uncertain within the mean free path  $\lambda$  and the thermal velocity  $c$ , respectively. It follows that both the assumptions are implicitly present in the Navier-Stokes equation [22]. Therefore, although the hydrodynamic (Navier-Stokes) equations are incomparably rich in the description of the turbulent phenomena, because they are able to take into account the specific conditions under which the transition takes place (the inlet conditions, flow environment, etc.) and give a dynamic picture of the transition, the driving force for the transition can have the same statistical origin as in the present model.

#### 4 Conclusions

A model for turbulent fluid flow has been proposed that considers the flow as a collection of localized spatial structures (eddies) and estimates the statistical weights of these collections for different Reynolds numbers. The Reynolds number is associated with the ratio between the total phase volume for the system and that for an elementary cell in which the state of the particles is uncertain. It has been found that i) the model successfully explains the onset of turbulence as well as some other characteristic properties of turbulent flows, and ii) the basic assumptions underlying the model, i.e. that the particles are identical and an elementary cell of the phase space exists where the state of the particles is uncertain, are implicitly present in the Navier-Stokes equation. Considered together, these findings suggest that with all the variety of dynamic phenomena that the hydrodynamic (Navier-Stokes) equations describe [5], the driving force for the turbulent transition is basically the same as in the present model, i.e. the system tends to occupy a spatial state that is statistically dominant at a given Reynolds number. The instability of the flow can then be a mechanism to initiate the structural rearrangement of the flow to find this state.

The present statistical aspect of the turbulent transition is believed to be important for a more comprehensive understanding of the nature of this challenging phenomenon as well as of the other related problems in the field of pattern formation [23].

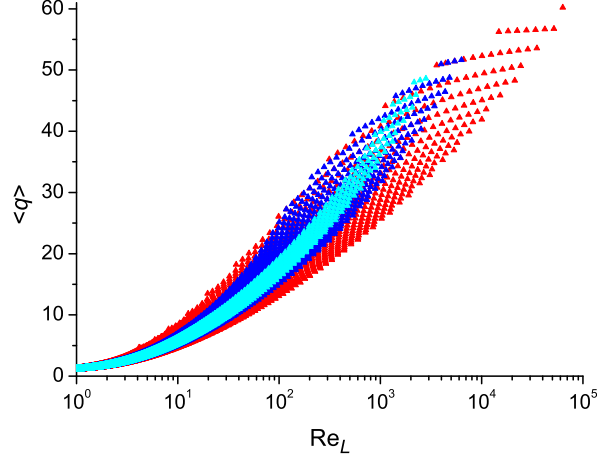
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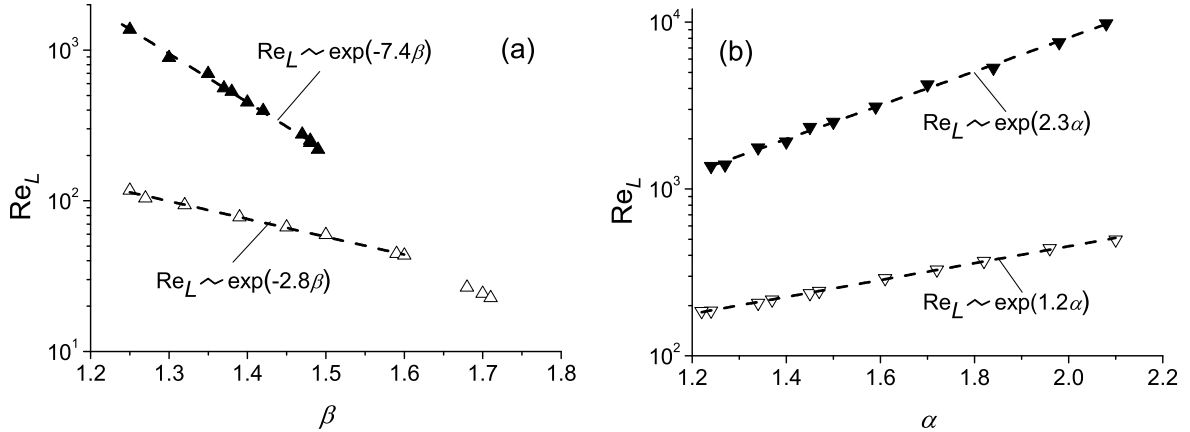
### Supplementary Material:

**Variation of the transition region with parameters  $\alpha$  and  $\beta$ .** Figure S1 illustrates how the distribution of Fig. 1 at  $q_{\max} = 200$  changes with variation of  $\alpha$  and  $\beta$ . The lower boundary shifts to the smaller values of  $Re_L$ , and the upper boundary to higher values of  $Re_L$ .



**Fig. S1** Average number of the elementary cells in the localized structures  $\langle q \rangle$  as a function of  $Re_L$ ;  $1 \leq q_i \leq 200$ . The cyan triangles are for  $-1.3 \leq \alpha \leq 1.3$  and  $-1.3 \leq \beta \leq 1.3$ , the blue triangles are for  $-1.5 \leq \alpha \leq 1.5$  and  $-1.5 \leq \beta \leq 1.5$  (as in Fig. 1), and the red triangles are for  $-2.0 \leq \alpha \leq 2.0$  and  $-2.0 \leq \beta \leq 2.0$ .

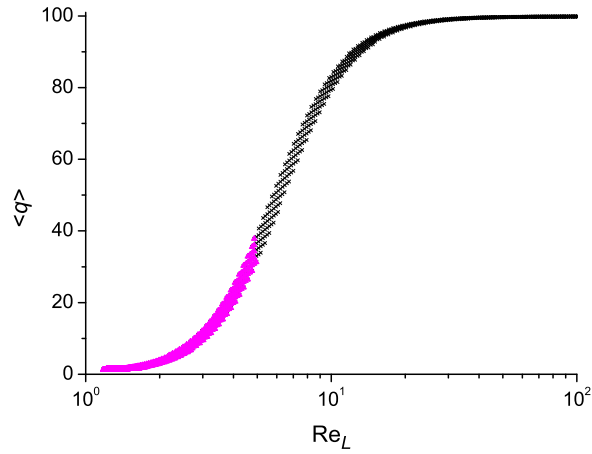
**Variation of the bounds of the flow stability with the Reynolds number.** Figure S2 shows how the Reynolds numbers at the lower (a) and upper (b) boundaries of the transition range change with  $\alpha$  and  $\beta$ . While the upper boundary monotonically shifts to larger values of  $Re_L$  as  $\alpha$  increases, the lower boundary shifts to smaller values of  $Re_L$  until  $\beta = \beta^*$  is reached ( $\beta^* \approx 1.725$  at  $\langle q \rangle = 20$ , and  $\beta^* \approx 1.5$  at  $\langle q \rangle = 40$ ). At larger values of  $\beta$  the lower boundary "freezes", as, e.g., it is seen from Fig. S1.



**Fig. S2** The Reynolds number  $Re_L$  corresponding to the lower (panel a) and the upper (panel b) boundaries of the transition range as a function of parameters  $\beta$  and  $\alpha$ , respectively. The empty and solid triangles are for  $\langle q \rangle = 20$  and  $\langle q \rangle = 40$ , respectively. The dashed lines show the best exponential fits to the data.

**Quasi-solid rotation of the localized structures.** If a quasi-solid rotation of the localized structures is supposed, in which case  $E_i \sim N_i^{5/3}$ ,  $\langle q \rangle$  rapidly increases with  $Re_L$  as previously, i.e. the





**Fig. S3** Average number of the elementary cells in the localized structures  $\langle q \rangle$  as a function of  $\text{Re}_L$  for quasi-solid rotation of the structures.  $1 \leq q_i \leq 100$ , and  $-0.25 \leq \alpha \leq 0.25$  and  $-0.25 \leq \beta \leq 0.25$ . Crosses represent all combinations of  $\alpha$  and  $\beta$ , and magenta triangles stand for the combinations which lead to the physically reasonable bell-shaped distribution.

statistical preference of the structured flow is preserved (Fig. S3). The difference is that the transition region is located within unreasonably low  $\text{Re}$  numbers. This is because the kinetic energy of the eddies is highly overestimated due to the absence of the decrease of the velocity outside the eddy core, which is characteristic of more realistic models, e.g. the Rankine and Lamb-Oseen vortices [1].

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